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HW 3

Math 575B
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1 Part II: Theory.

1. Derive the Modified Euler's Method:

$$x(t+h) = x(t) + hf(t + \frac{h}{2}, x(t) + \frac{h}{2} f(t, x(t))) \tag{1}$$

Solution. Find the truncation error of Explicit Euler (EE).

$$x(t+h) = x(t) + hf(t, x(t)) + T \tag{2}$$

Note that $x' = f(t, x(t))$, Taylor expand $x(t+h)$ about the point t:

$$\begin{aligned} x(t+h) &= x(t) + hx'(t) + \frac{h}{2} x''(t) + \mathcal{O}(h^3) && (3) \\ &= x(t) + hf(t, x(t)) + \frac{h^2}{2} f'(t, x(t)) + \mathcal{O}(h^3) && (4) \end{aligned}$$

The truncation error is the Taylor expansion of $x(t+h)$ minus the EE step:

$$\begin{aligned} T &= x(t+h) - (x(t) + hf(t, x(t))) \\ &= x(t) + hf(t, x(t)) + \frac{h^2}{2} f'(t, x(t)) - x(t) - hf(t, x(t)) + \mathcal{O}(h^3) \\ &= \frac{h^2}{2} f'(t, x(t)) + \mathcal{O}(h^3) \end{aligned}$$

Plug into 2:

$$x(t+h) = x(t) + hf(t, x(t)) + \frac{h^2}{2} f'(t, x(t)) + \mathcal{O}(h^3)$$

Take expansion of $x(t + \frac{h}{2})$:

$$\begin{aligned} x(t + \tfrac{h}{2}) &= x(t) + \frac{h}{2} f(t, x(t)) + \frac{h^2}{8} f'(t, x(t)) + \mathcal{O}(h^3), \text{ and after another half step} \\ x(t+h) &= x(t + \tfrac{h}{2}) + \frac{h}{2} f(t + \tfrac{h}{2}, x(t + \tfrac{h}{2})) + \frac{h^2}{8} f'(t + \tfrac{h}{2}, x(t + \tfrac{h}{2})) + \mathcal{O}(h^3) \end{aligned}$$

Substitute for $x(t + \frac{h}{2})$ then

$$x(t+h) = x(t) + \frac{h}{2} f(t, x(t)) + \frac{h}{2} f(t + \tfrac{h}{2}, x(t + \tfrac{h}{2})) \tag{5}$$

$$+\frac{h^2}{8}\left(f'(t,x(t))+f'(t+\tfrac{h}{2},x(t+\tfrac{h}{2}))\right)+\mathcal{O}(h^3)$$

If we note here that, given that $f':\mathbb{R}^2\to\mathbb{R}$ is continuous, then by the Intermediate Value Theorem, there exists some k where $f'(t,x(t))\le k\le f'(t+\frac{h}{2},x(t+\frac{h}{2}))$. Let

$$k=\frac{f'(t,x(t))+f'(t+\frac{h}{2},x(t+\frac{h}{2}))}{2}$$

then $\exists(\tau,\zeta)$ where $\tau\in[t,t+\frac{h}{2}]$ and $\zeta\in[x(t),x(t+\frac{h}{2})]$[1] such that $f'(\tau,\zeta)=k$. We now rewrite Eqn 5 as

$$x(t+h)=x(t)+\frac{h}{2}f(t,x(t))+\frac{h}{2}f(t+\tfrac{h}{2},x(t+\tfrac{h}{2}))+\frac{h^2}{4}f'(\tau,\zeta)+\mathcal{O}(h^3) \tag{6}$$

Subtract Eqn 4 from 2×Eqn 6:

$$2(\text{Eqn }4)-\text{Eqn }2=x(t)+hf(t+\tfrac{h}{2},x(t+\tfrac{h}{2}))+\frac{h^2}{2}f'(\tau,\zeta)-\frac{h^2}{2}f'(t,x(t))+\mathcal{O}(h^4)$$

$f':\mathbb{R}^2\to\mathbb{R}$ is a continuous function, and also assume that it is differentiable on $[(t,x(t)),(\tau,\zeta)]$[2]. By the Mean Value Theorem, there exists at least one point $(\lambda,\xi)\in[(t,x(t)),(\tau,\zeta)]$ such that

$$f''(\lambda,\xi)=\frac{f'(\tau,\zeta)-f'(t,x(t))}{h} \tag{7}$$

which we use to rewrite our previous equation as

$$x(t+h)=x(t)+hf(t+\tfrac{h}{2},x(t)+\tfrac{h}{2}f(t,x(t)))+\frac{h^3}{2}f''(\lambda,\xi)+\mathcal{O}(h^3) \tag{8}$$

Where we again substitute for $x(t+\frac{h}{2})$. Therefore we have Derived the Modified Euler Method, Eqn 8, which has truncation error $\mathcal{O}(h^3)$.

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[1] If $x(t+\frac{h}{2})\le x(t)$ this result will follow similarly.

[2] WLOG $[(\tau,\zeta),(t,x(t))]$

2. Prove the following theorem:

Theorem 1. *Suppose $y(x)$ has three continuous and bounded derivatives. It is also the solution to $y' = f(x, y)$, with $y(a) = y_0$. Let f_{yy} be bounded and continuous, and let the initial error e_0 in the approximate solution u_i to $y(x_i)$ be $e_0 = \xi_0 h$, where ξ_0 is independent of h. Then $e_j = h\xi(x_j) + \mathcal{O}(h^2) \quad j = 0, 1, \ldots, N$, where $\xi(x)$ is a solution of the linear problem*

$$\begin{aligned} \frac{d\xi}{dx} &= f_y(x, y(x))\xi - \frac{1}{2}y''(x), & (9) \\ \xi(a) &= \xi_a. & (10) \end{aligned}$$

Proof. Write $y_j = u_j + e_j$ and take an EE step:

$$y_{j+1} = u_j + e_j + hf(x_j, u_j + e_j) + \mathcal{O}(h^2) \tag{11}$$

Taylor expand the right side and make use of the definition of y_j:

$$u_{j+1} + e_{j+1} = u_j + e_j + hf(x_j, u_j) + he_j f_y(x_j, u_j) + \frac{(he_j)^2}{2} f_{yy}(x_j, u_j) + \mathcal{O}(h^3) \tag{12}$$

Apply definition of ξ_j:

$$u_{j+1} + \xi_{j+1}h = u_j + \xi_j h + hf(x_j, u_j) + h^2\xi_j f_y(x_j, u_j) + \mathcal{O}(h^3) \tag{13}$$

Rearrange the terms:

$$u_{j+1} - u_j - hf(x_j, u_j) = -\xi_{j+1}h + \xi_j h + h^2\xi_j f_y(x_j, u_j) + \mathcal{O}(h^3) \tag{14}$$

The first two terms on the right side, after dividing through by h^2, of 14 combine to make ξ'. Rearranging and combining like terms yields

$$\begin{aligned} \xi_j' &= \frac{-u_{j+1} + u_j + hf(x_j, u_j)}{h^2} + \xi_j f_y(x_j, u_j) \\ &= \frac{\frac{-(u_{j+1} - u_j)}{h} - \frac{(u_j - u_{j-1})}{h}}{h} + \xi_j f_y(x_j, y_j) \\ &= -\frac{1}{2}y_j'' + \xi_j f_y(x_j, y_j) \end{aligned}$$

Thus, we have reached the ending point. □

2 Part III: Experiment.

1. Solve IVP:

$$y' = 2xy \tag{15}$$

for $y(0) = 1$ and $x \in [0, 3]$

Solution. a.) Using ERK4 with varying time steps, it was shown that a fixed-time step algorithm was insufficient for solving the problem. The solution cannot be approximated with ERK4; the error term doesn't converge. When the step size is too large, the algorithm tends to overshoot the exact solution; whereas for a sufficiently small time step, the algorithm undershoots. An adaptive step size algorithm is needed to optimize the numerical solution to the exact.
b.) Using the built-in Matlab command ode45, the numerical solution converged to the exact solution, and in a very small number of integration steps (about 45).

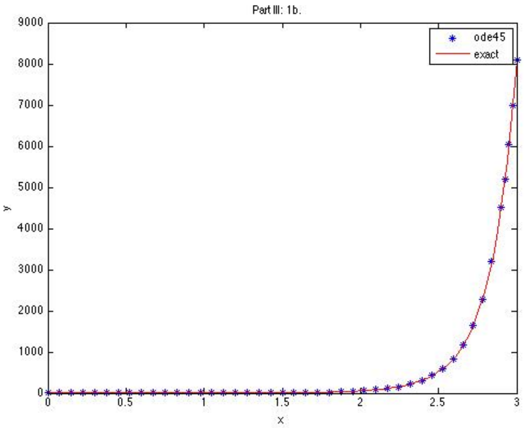

Figure 1: Here we see a plot of y vs. x for the ERK4 approximation to $y' = 2xy$

c.) Code was written to do adaptive time step integration. Resulting plots are:

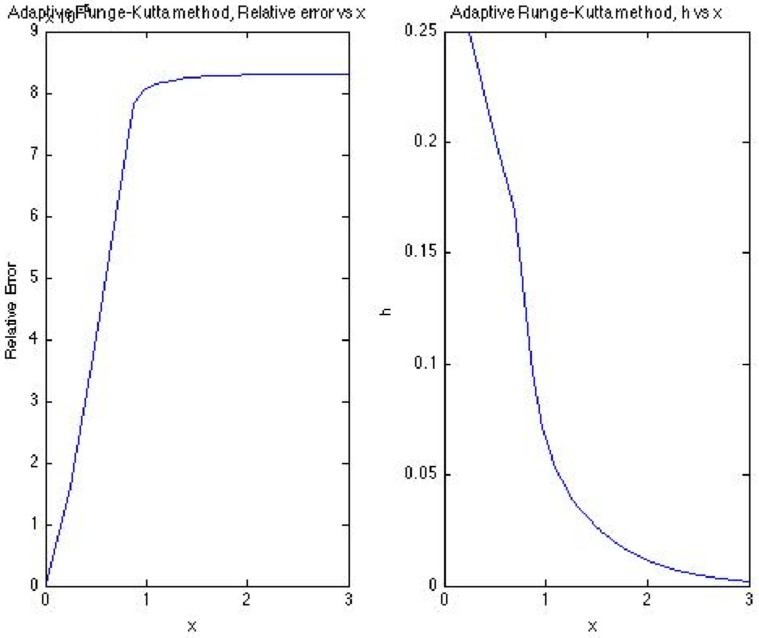

Figure 2: h vs. x and Rel. Err. vs. x

It would be unwise to plot the approximate and the exact solutions because you will not be able to nail down where a potential problem in your code came from. Also, the scales for y in the graph are very large, and it would be difficult to resolve.

2. Look at case where $y' = f(x, y)$ is not smooth.

   Solution. First, it should be noted that the problem point comes at $x = 1$, because $y' = f(x, y)$ has a kink in it, i.e. it is not differentiable.
   a.) (code written)

b.) $h \rightarrow 0$ as $x \rightarrow 0$, and during this time, the relative error spikes up at the discontinuity:

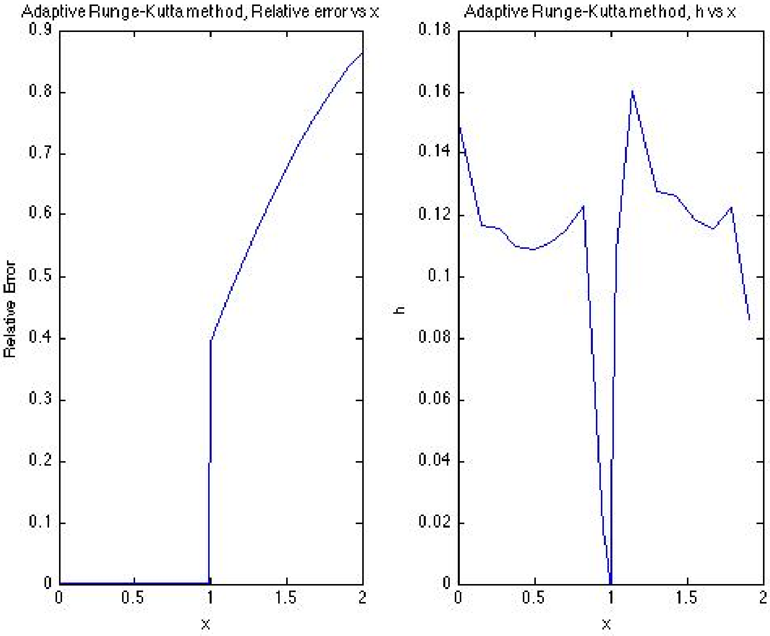

Figure 3: h vs. x and Rel. Err. vs. x

c.) The above example encapsulates this part too. The TOL, h_{min}, and h_{max} are the same.

d.) At $x = 2$, the solution is very close to e, as expected. I found that looking in a small ϵ-window around 1 (.01 in either direction), TOL can be relaxed in order to minimize how small the step size gets. Therefore the code doesn't stall and the desired result is achieved.

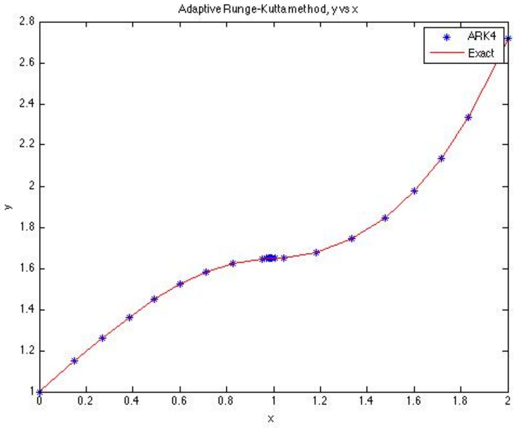

Figure 4: Numerical vs. Exact solution